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Note

A Note on Derivatives of Bernstein Polynomials*

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In this note we improve two results on derivatives of Bernstein polynomials and smoothness of functions.

The Bernstein polynomials on $C[0, 1]$ are given by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, P_{n,k}(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

In 1985, Z. Ditzian [1] proved for $f \in C[0, 1]$, $r = 1, 2$, $0 < \alpha < r$ that under the assumption $\omega_r(f, t) = O(t^\beta)$ for some $\beta > 0$

$$|B_n^{(r)}(f, x)| \leqslant M_f(\min\{n^2, n/(x(1-x))\})^{(r-\alpha)/2} \Leftrightarrow \omega_r(f, t) = O(t^\alpha), \tag{2}$$

where for $r \in N$

$\omega_r(f, t) = \sup_{0 < h \leqslant t} \|\Delta_h^r f\|_\infty$,

$\Delta_h^r f(x) = \sum_{k=0}^{r} \binom{r}{k}(-1)^k f(x + (r/2 - k)h)$, if $[x - rh/2,\ x + rh/2] \subset [0, 1]$;

$\Delta_h^r f(x) = 0$, otherwise.

Recently, the author [4] extended this result to higher levels of smoothness as

$$|B_n^{(r)}(f, x)| \leqslant M_f(\min\{n^2, n/(x(1-x))\})^{(r-\alpha)/2} \Leftrightarrow \omega_r(f, t) = O(t^\alpha) \tag{3}$$

for $r \in N$, $0 < \alpha < r$, under the assumption $\omega_r(f, t) = O(t^\beta)$ for some $\beta > 0$.

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In this note we improve these two results as follows. Some ideas are from [1, 3].

THEOREM 1. *Suppose that* $f \in C[0,1]$ *satisfies* $\omega_r(f,t)=O(t^\beta)$ *for some* $\beta>0$, $r \in N$, *then for* $0<\alpha<r$ *we have*

$$|B_n^{(r)}(f,x)| \leqslant M_f(n/(x(1-x)))^{(r-\alpha)/2} \Leftrightarrow \omega_r(f,t)=O(t^\alpha). \tag{4}$$

Proof. By [4] it is sufficient for us to prove the inverse part.

Let $B_n(f,r,x)$ be the combination of Bernstein polynomials defined as

$$B_n(f,r,x)=\sum_{i=0}^{r-1} C_i(n)\, B_{n_i}(f,x), \tag{5}$$

where $n_i \in N$ and $C_i(n)$ satisfy with an absolute constant $C \in N$

$$\begin{aligned} &(a)\quad n=n_0<\cdots<n_{r-1}\leqslant Cn,\\ &(b)\quad \sum_{i=0}^{r-1}|C_i(n)|\leqslant C,\\ &(c)\quad \sum_{i=0}^{r-1} C_i(n)=1,\\ &(d)\quad \sum_{i=0}^{r-1} C_i(n)\, n_i^{-k}=0, \qquad \text{for}\quad k=1,\cdots,r-1. \end{aligned} \tag{6}$$

For these operators we have (see [4])

$$|B_n(f,r,x)-f(x)| \leqslant M_0\omega_r(f,\max\{1/n,\sqrt{x(1-x)/n}\}) \tag{7}$$

with a positive constant M_0 independent of $n \in N$ and $x \in [0,1]$.

Let $0<h\leqslant t\leqslant 1/(8r)$, $[x-rh/2,\, x+rh/2]\subset(0,1)$, $n\in N$. Denote $\varphi(x)=x(1-x)$, $d_1(n,x,h)=\max_{0\leqslant k\leqslant r}\{(\varphi(x+(r/2-k)\,h)/n)^{1/2}\}$, $d(n,x,h)=\max\{1/n,\, d_1(n,x,h)\}$. Then we have

$$\begin{aligned} |\Delta_h^r f(x)| &\leqslant |\Delta_h^r(f-B_n(f,r,\cdot))(x)| \\ &\quad +\int\cdots\int_{-h/2}^{h/2}\left|B_n^{(r)}\left(f,r,x+\sum_{j=1}^r y_j\right)\right| dy_1\cdots dy_r \\ &\leqslant 2^r M_0\omega_r(f,d(n,x,h))+\sum_{i=0}^{r-1}|C_i(n)|\,M_f \\ &\quad\times\int\cdots\int_{-h/2}^{h/2}\left(n_i\Big/\varphi\left(x+\sum_{j=1}^r y_j\right)\right)^{(r-\alpha)/2} dy_1\cdots dy_r \\ &\leqslant 2^r M_0\omega_r(f,d(n,x,h))+C^{r+1}M_f n^{(r-\alpha)/2}h^\alpha \\ &\quad\times\left(\int\cdots\int_{-h/2}^{h/2}\left(\varphi\left(x+\sum_{j=1}^r y_j\right)\right)^{-r/2} dy_1\cdots dy_r\right)^{(r-\alpha)/r} \\ &\leqslant 2^r M_0\omega_r(f,d(n,x,h))+C^{r+1}M_f(C_r+1) \\ &\quad\times h^r(\max_{0\leqslant k\leqslant r}\{(\varphi(x+(r/2-k)\,h)/n)^{1/2}\})^{(\alpha-r)}. \end{aligned} \tag{8}$$

Here we have used the following estimate in [4]

$$\int \cdots \int_{-h/2}^{h/2} \left(\varphi\left(x + \sum_{j=1}^{r} y_j\right)\right)^{-r/2} dy_1 \cdots dy_r$$
$$\leqslant C_r\big(\max_{0 \leqslant k \leqslant r} \{\varphi(x + (r/2 - k)\, h)\}\big)^{-r/2} h^r \qquad (9)$$

with a constant $C_r > 0$ depending only on r.

Note that $d(n+1, x, h) < d(n, x, h) < 2d(n+1, x, h)$. For any $\delta \in (0, 1/(8r)]$ we can choose $n \in N$ such that $d(n, x, h) \leqslant \delta < 2d(n, x, h)$.

If $d_1(n, x, h) \geqslant 1/n$, we have from (8)

$$|\Delta_h^r f(x)| \leqslant 2^r M_0 \omega_r(f, \delta) + 2^r C^{r+1} M_f (C_r + 1)\, h^r \delta^{\alpha - r}.$$

If $d_1(n, x, h) < 1/n$, we have $n = 1/d(n, x, h) \leqslant 2/\delta$. Therefore we also have from (8)

$$|\Delta_h^r f(x)| \leqslant 2^r M_0 \omega_r(f, \delta) + C^{r+1} M_f (C_r + 1)\, h^r n^{(r-\alpha)/2} (\varphi(h/2))^{(\alpha - r)/2}$$
$$\leqslant 2^r M_0 \omega_r(f, \delta) + 4^r C^{r+1} M_f (C_r + 1)\, h^{(r+\alpha)/2} \delta^{(\alpha - r)/2}.$$

Combining the above two cases we obtain

$$|\Delta_h^r f(x)| \leqslant M_1 \omega_r(f, \delta) + M_2 (t^r \delta^{\alpha - r} + t^{(r+\alpha)/2} \delta^{(\alpha - r)/2}),$$

where the constants $M_1, M_2 > 1$ are independent of x, h, t, and δ, which implies

$$\omega_r(f, t) \leqslant M_1 \omega_r(f, \delta) + M_2 (t^r \delta^{\alpha - r} + t^{(r+\alpha)/2} \delta^{(\alpha - r)/2}).$$

Suppose that $\omega_r(f, h) \leqslant M_3 h^{\beta}$. Let $A = (2M_1)^{1/\alpha + 1/\beta}$, $\delta = t/A$. We then have by induction

$$\omega_r(f, t) \leqslant M_1 \omega_r(f, t/A) + 2M_2 A^{r-\alpha} t^{\alpha}$$
$$\leqslant \cdots$$
$$\leqslant M_1^m \omega_r(f, tA^{-m}) + 2M_2 A^{r-\alpha} t^{\alpha} \sum_{k=0}^{m-1} (M_1 A^{-\alpha})^k$$
$$\leqslant M_3 M_1^m t^{\beta} A^{-m\beta} + 2M_2 A^r t^{\alpha} (A^{\alpha} - M_1)^{-1}$$
$$\leqslant M_3 t^{\beta} 2^{-m} + 2M_2 A^r t^{\alpha}.$$

By letting $m \to \infty$, we have

$$\omega_r(f, t) \leqslant 2M_2 A^r t^{\alpha}.$$

Our proof is now complete.

By the same method as in [1], we can prove a simpler result for Kantorovich operators

$$K_n(f, x) = \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt\, P_{n,k}(x). \tag{10}$$

THEOREM 2. *For $r \in N$, $f \in C[0, 1]$, $0 < \alpha < r$, we have*

$$|K_n^{(r)}(f, x)| \leqslant M_f'(n/(x(1-x)))^{(r-\alpha)/2} \Leftrightarrow \omega_r(f, t) = O(t^\alpha).$$

Remark. A similar result holds for Bernstein–Durrmeyer operators [2].

Remark. I conjecture that a similar improvement can be given for algebraic polynomials of best approximation.

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REFERENCES

1. Z. DITZIAN, Derivatives of Bernstein polynomials and smoothness, *Proc. Amer. Math. Soc.* **93** (1985), 25–31.
2. Z. DITZIAN AND K. IVANOV, Bernstein type operators and their derivatives, *J. Approx. Theory* **56** (1989), 72–90.
3. Z. DITZIAN AND V. TOTIK, Moduli of smoothness, *in* "Springer Series in Computational Mathematics," Vol. 9, Springer-Verlag, Berlin/Heidelberg/New York, 1987.
4. DING-XUAN ZHOU, On smoothness characterized by Bernstein type operators, submitted for publication.